Information advantages in hiring under a budget constraint: weak convergence comparisons

Frans A. Boshuizen *
Econometric Institute, Erasmus Universiteit Rotterdam, The Netherlands

Robert P. Kertz †
School of Mathematics, Georgia Institute of Technology Atlanta, Georgia

October 6, 1995

Abstract

Let $X_1, X_2, \cdots$ be a sequence of positive i.i.d. random variables with distribution function $F(x)$, and let $X_1^{(n)} \leq X_2^{(n)} \leq \cdots \leq X_n^{(n)}$ be the sequence of order statistics of $X_1, \ldots, X_n$. For a sequence $(c_n)_{n\geq 1}$ of positive constants the off-line minimal interview count random variable is defined by $M^e(c_n) := \min\{j : j \geq n : X_1^{(j)} + \cdots + X_n^{(j)} \leq c_n\}$. In this paper, an asymptotic joint distributional comparison is given between the off-line count $M^e(c_n)$ and on-line interview counts M_n^τ for 'good' sequential (on-line) policies τ satisfying the sum constraint $\sum_{i=1}^n X_{\tau_i} \leq c_n$. Specifically, for such policies τ, under appropriate conditions on the distribution function $F(x)$ and the constants $(c_n)_{n\geq 1}$, we find sequences of positive constants $(\delta_n)_{n\geq 1}$, $(\Delta_n)_{n\geq 1}$ and $(\Delta_n')_{n\geq 1}$ such that

$$\left(\Delta_n\left(\frac{M^e(c_n)}{\delta_n}-1\right), \Delta_n'\left(\frac{M_n^\tau}{\delta_n}-1\right)\right) \Rightarrow (W, W') \text{ as } n \to \infty,$$

for some nondegenerate r.v.'s W and W'.
Abbreviated Title. Joint convergence of interview counts

1 Introduction

In this paper we consider the following hiring problem introduced by Chen et al. [2]. A company wants to hire n individuals whose total salary demand can not exceed a predetermined budget c_n. The objective of the company is to minimize the number of interviews that have to be made

*Research partly supported by a Fulbright Grant

†Corresponding author; Research partly supported by National Science Foundation Grant DMS 92-09586

AMS 1991 subject classifications. Primary 60G40, 60G70; secondary 60F05, 60F15, 60G35, 90B50.

Key words and phrases. Joint convergence in distribution, strong approximation, extreme sums, order statistics, Brownian bridge, counting r.v.'s, best choice problems, threshold strategies

in order to hire those n individuals. Let the r.v.'s $X_1, X_2, \cdots$ denote the salary demands of the applicants; these are assumed to be positive i.i.d. r.v.'s each with d.f. $F(x)$. Let n be any positive integer and c_n be any positive real number, and define

$$M^e(c_n) = M_n^e(c_n) = \min\{j : j \geq n \text{ and } X_1^{(j)} + \cdots + X_n^{(j)} \leq c_n\} \tag{1}$$

if this set is nonempty and $= \infty$ otherwise. Then $M^e(c_n)$ is the minimal number of interviews necessary to obtain n individuals whose total salary demand is $\leq c_n$, when the random salary demands of the persons interviewed are sampled from distribution F. This is the smallest interview count, obtainable by the 'prophetic' company using an off-line strategy, that is, under knowledge of the exact salary demands and with no order restrictions on offers made to those interviewed. This off-line, smallest interview-count strategy is denoted τ^e.

For this problem with fixed n, define a policy τ to be any sequence of stopping times $\tau_1 < \cdots < \tau_n$ (for $(X_j)_{j\geq 1}$). For n applicants, the counting r.v. associated with policy $\tau = (\tau_1, \ldots, \tau_n)$ is defined by $M_n^\tau := \tau_n$. For any policy $\tau = (\tau_1, \ldots, \tau_n)$ for which the sum constraint $\sum_{i=1}^n X_{\tau_i} \leq c_n$ is satisfied, M_n^τ has interpretation as the number of persons interviewed in order to hire n individuals with total salary demand under the budget constraint c_n, when applicants are interviewed in a specific order, salary demands become known first at the interview, and the decision to hire is made at the interview according to policy τ, without recall possible. Thus M_n^τ is an on-line interview count, an interview count determined under an on-line policy.

In this paper, off-line interview count $M^e(c_n)$ is compared to on-line interview count M_n^τ for 'good' policies τ satisfying the budget (sum) constraint. This is a 'dual' problem to the 'primal' smallest fit problem studied by Coffman, Flatto and Weber [5] (see also Boshuizen and Kertz [1]); this 'duality' is present both in the problem formulation and in the mathematical analysis of the problem. One must choose some criteria for this comparison, and then choose a 'good' on-line policy for hiring applicants under this criteria. Under an expectation-based criteria, the company wants to minimize the expected number of interviews in order to hire n individuals over on-line policies satisfying the budget (sum) constraint, and a 'best' policy yields $EM_n^*(c_n)$, where *optimal (interview) counting r.v.* $M_n^*(c_n)$ is any r.v. satisfying

$$EM_n^*(c_n) = \inf\left\{EM_n^\tau : \tau = (\tau_1, \ldots, \tau_n) \text{ is a policy such that } \textstyle\sum_{i=1}^n X_{\tau_i} \leq c_n\right\}.$$

An optimal policy τ^* for this expectation-based criteria is described in Chen et al. [2], and the structure of τ^* is similar to the optimal policy in the smallest fit (primal) problem discussed in Coffman et al. [5]. As mentioned in [2], explicit calculation and direct analysis using τ^* is complicated and one would like to have a simpler form of policy, such as a stopped threshold policy, with a 'good' asymptotic behavior.

For a sequence of positive constants $(c_n)_{n\geq 1}$ and for n fixed, a *two-stage threshold policy* π_n is a policy of the following form. Let $(\varepsilon_n)_{n\geq 1}$ be positive constants and let the sequence $(\tilde{c}_n)_{n\geq 1}$ satisfy $0 < \tilde{c}_n < c_n$. From the group of applicants, first accept those individuals whose salary demand is $\leq \varepsilon_n$ and select these only when the sum of the salaries of those selected is $\leq \tilde{c}_n$. From the point that this sum of salaries of those selected would exceed the budget constraint $\tilde{c}_n$, select only individuals whose salary demand is $\leq (c_n - \tilde{c}_n)/n$ until a total of n applicants is selected. Note that this two-stage policy π_n always selects n individuals whose total salary demand is $\leq c_n$, since after the first stage the budget is still $\geq c_n - \tilde{c}_n$ and $n \times (c_n - \tilde{c}_n)/n = c_n - \tilde{c}_n$. The interview counting r.v. associated with the policy π_n is denoted by $M^{\pi_n}(c_n) = M^{\pi_n}(\varepsilon_n, \tilde{c}_n, c_n) := M_n^{\pi_n}$.

A more precise description is given of the two-stage threshold policy π_n and its associated counting r.v. $M^{\pi_n}(c_n)$, for use in the analysis of this paper. For the first stage, define r.v.'s $k_1, k_2, \ldots$ and $W_1, W_2, \ldots$ as follows. Let $k_0 := 0$, and for $i = 1, 2, \ldots$, define

$$k_i = k_{i,n} := \inf\{j \geq 1 : X_{j+k_0+\cdots+k_{i-1}} \leq \varepsilon_n\} \text{ and } W_i = W_{i,n} := X_{k_1+\cdots+k_i}. \tag{2}$$

For the second stage, define r.v.'s $\eta, J, k_1', k_2', \ldots$ and $W_1', W_2', \ldots$ as follows. Define

$$\begin{aligned} \eta &= \eta_n := \min\{j : 1 \leq j \leq n \text{ and } W_1 + \cdots + W_j > \tilde{c}_n\} \\ &\quad\ \text{if this set is nonempty and } = n+1 \text{ otherwise, and} \\ J &= J_n := k_0 + \cdots + k_\eta. \end{aligned} \tag{3}$$

Let $k_0' := 0$, and for $i = 1, 2, \ldots$, define

$$k_i' = k_{i,n}' := \inf\{j \geq 1 : X_{j+J+k_0'+\cdots+k_{i-1}'} \leq (c_n - \tilde{c}_n)/n\} \text{ and } W_i' := X_{J+k_1'+\cdots+k_i'}. \tag{4}$$

These r.v.'s are defined on a set of probability one. Policy π_n is defined through random times $(\pi_{j,n})_{1\leq j\leq n}$ given for $j = 1, \ldots, n$ by

$$\begin{aligned} \pi_{j,n} &= \left(\sum_{i=1}^{j} k_i\right) I(j < \eta) + \left(\sum_{i=1}^{\eta} k_i + \sum_{i=1}^{j+1-\eta} k_i'\right) I(\eta \leq j), \text{ and} \\ M^{\pi_n}(c_n) &= \pi_{n,n} = \sum_{i=1}^{\eta\wedge n} k_i + \sum_{i=1}^{n+1-\eta} k_i' \end{aligned} \tag{5}$$

(where $\sum_{i=1}^{0} k_i' := 0$). The following lemma summarizes properties of these objects. Its proof follows from elementary properties of independent r.v.'s and is thus omitted.

Lemma 1.1 *Let $n \geq 1$ be given.*
(i) $(k_i)_{i\geq 1}$ are i.i.d. r.v.'s, each geometric $(F(\varepsilon_n))$-distributed; $(W_i)_{i\geq 1}$ are i.i.d. r.v.'s, each with d.f. $F_W(x) := F(x)/F(\varepsilon_n)$ for $0 \leq x \leq \varepsilon_n$; and $(k_i)_{i\geq 1}$ and $(W_i)_{i\geq 1}$ are independent r.v.'s.
(ii) Random variable η is a stopping time with respect to $(W_i)_{i\geq 1}$, and η is independent of $(k_i)_{i\geq 1}$.
(iii) $(k_i')_{i\geq 1}$ are i.i.d. r.v.'s, each geometric $(F((c_n - \tilde{c}_n)/n))$-distributed; and $(k_i')_{i\geq 1}$ is independent of $\{(k_i)_{1\leq i\leq \eta}, (W_i)_{1\leq i\leq \eta}, \eta\}$.
(iv) Random times $\pi_{j,n}$, $1 \leq j \leq n$, are stopping times for $(X_i)_{i\geq 1}$ satisfying $\pi_{1,n} < \cdots < \pi_{n,n}$.

It follows from Lemma 1.1 *(iv)* that the two-stage threshold policy π_n is indeed a policy.

In the paper by Coffman et al. [5], it was shown that under the hypotheses $c_n \equiv c$, $\tilde{c}_n \equiv \tilde{c}$, for some $0 < \tilde{c} < c$, $\varepsilon_n = \varepsilon = \nu^{\leftarrow}(\tilde{\tilde{c}}/n)$ with $\nu(x) = E(X_1|X_1 \leq x)$ and $0 < \tilde{\tilde{c}} < \tilde{c} < c$, and d.f. F of X_1 is continuous and strictly increasing on its support, and $F(x) \approx Ax^\alpha$ as $x \downarrow 0$ for some $A, \alpha > 0$, then $\limsup_{\tilde{\tilde{c}}\to c} \limsup_{n\to\infty} EM_n^{\pi_n}(\varepsilon, \tilde{c}, c)/EM_n^e(c) \leq 1$ (which in turn implies $\lim_{n\to\infty} EM_n^*(c)/EM_n^e(c) = 1$). In the case $c_n = \alpha n$, $0 < \alpha < EX_1$, it was proved by Chen et al. [2] that $\lim_{n\to\infty} E\tilde{M}_n(c_n)/EM_n^e(c_n) = 1$ where $\tilde{M}_n(c_n)$ is a counting r.v. using a one-stage policy with threshold $\nu^{\leftarrow}(\alpha)$, again under the hypothesis that F is continuous and strictly increasing on its support.

In this paper, precise asymptotic joint distributional comparisons are made of the off-line interview count $M^e(c_n)$ and on-line interview counts M_n^τ for 'good' policies τ satisfying the sum constraint, through the following definition. A sequence of policies $(\tau^n)_{n\geq 1}$ satisfying the sum constraint $\sum_{i=1}^n X_{\tau_i^n} \leq c_n$ for $n \geq 1$ is said to be a **consistent approximator** of the off-line, smallest interview-count strategy τ^e if there exists positive constants $(\delta_n)_{n\geq 1}$, $(\Delta_n)_{n\geq 1}$ and $(\Delta'_n)_{n\geq 1}$ such that

$$\left(\Delta_n\left(\frac{M^e(c_n)}{\delta_n}-1\right),\, \Delta'_n\left(\frac{M_n^{\tau^n}}{\delta_n}-1\right)\right) \Rightarrow (W, W') \text{ as } n \to \infty \tag{6}$$

for some nondegenerate r.v.'s W and W'. This definition is analogous to the definition for 'good' policies in the 'primal' smallest-fit problem in [1]. In Theorems 2.2 and 4.1 it is shown that the sequence of two-stage threshold policies $(\pi_n)_{n\geq 1}$ is a consistent approximator of τ^e, for appropriate assumptions on the budgets $(c_n)_{n\geq 1}$ and distribution F, and for appropriately chosen sequences $(\tilde{c}_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$.

Throughout the paper the following notation is used. For a nondecreasing function h on a subset of $\mathbb{R}$, the left-continuous inverse of h is defined by $h^{\leftarrow}(s) = \inf\{x \in S : h(x) \geq s\}$. For a distribution function F, $l_F = \inf\{x : F(x) > 0\}$ denotes the left end point of the support of F. The notations $o_P(1)$ and $O_P(1)$ are used to denote sequences of r.v.'s which are respectively converging to zero in probability and bounded above and below by a finite constant uniformly for all n large. For two r.v.'s X and Y, $X \stackrel{\mathrm{d}}{=} Y$ if the distributions of X and Y are the same. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$ and $\lceil x \rceil$ denotes the smallest integer $\geq x$. For two sequences $(m_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ we write $m_n \approx l_n$ if $\lim_{n\to\infty} m_n/l_n = 1$. For two functions $f_1(s)$ and $f_2(s)$ on $(0,1)$ we write $f_1(s) \approx f_2(s)$ as $s \downarrow 0$ if $\lim_{s\downarrow 0} f_1(s)/f_2(s) = 1$.

2 Statement of main results for medium-sized constraints

In Sections 2 and 3, settings with 'medium-sized' budget constraints are considered in which appropriate normalizations of the pair of interview counting r.v.'s $M^e(c_n)$ and $M^{\pi_n}(c_n)$ converge weakly (as $n \to \infty$) to a pair of r.v.'s (Y_1, Y_2^+) where (Y_1, Y_2) has a multivariate normal distribution and $a^+ = \max\{a, 0\}$.

The settings considered are distribution functions $F(x)$ in Cases I and III for minima with $l_F = 0$. For $F(x)$ in Case I for minima with $l_F = 0$, define the auxiliary function $c(s)$ on the interval $(0,1)$ by $c(s) = s^{-1}\int_0^s u dF^{\leftarrow}(u)$; and for $F(x)$ in Case III for minima with $l_F = 0$ and index $\alpha > 0$, recall the representation $F^{\leftarrow}(s) = s^{-a}L(s)$ where $L(s)$ is a function on $(0,1)$ slowly varying at zero and $a = -1/\alpha$. See the Appendix for background results on $F^{\leftarrow}(s)$ and $c(s)$ used in this paper, and for additional results used from extreme value theory. Note that for these distribution functions, $F(0) = 0$.

For distribution function $F(x)$, define function $H(t)$ on $(0,1)$ by $H(t) := t^{-1}\int_0^t F^{\leftarrow}(s)ds$. Function H is continuous, and strictly increasing on $(l_F, 1)$. For $F(x)$ in Case I for minima with $l_F = 0$, $H(t) \approx F^{\leftarrow}(t)$ as $t \to 0$, and $H(t)$ is slowly varying at zero, and $c(t) = F^{\leftarrow}(t) - H(t)$ for $0 < t < 1$. For $F(x)$ in Case III for minima with $l_F = 0$ and with index $\alpha = -1/a > 0$, $H(t) \approx (1-a)^{-1}F^{\leftarrow}(t)$ as $t \to 0$ and $H(t)$ is regularly varying at zero with index $-a$; and $H^{\leftarrow}(t) \approx (1-a)^{-1/a}F(t)$ as $t \to 0$ and $H^{\leftarrow}(t)$ is regularly varying at zero with index $\alpha = -1/a$.

Throughout Sections 2 and 3, the sequence $(c_n)_{n\geq 1}$ of budget constraint parameters is assumed to be a sequence of positive constants satisfying $0 < c_n \leq n$ and $\lim_{n\to\infty} c_n/n = 0$. This is the 'medium-sized' budget constraint case, in contrast to the 'large-budget' case of Section 4.

Define auxiliary sequences $(\delta_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ of positive constants by

$$\delta_n := n/H^{\leftarrow}(c_n/n) \text{ and } \varepsilon_n := F^{\leftarrow}\left(H^{\leftarrow}(c_n/n)\right) \text{ for } n \geq 1, \tag{7}$$

so that $(\delta_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ satisfy $n/\delta_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, and $c_n/n = H(n/\delta_n)$ and $\varepsilon_n = F^{\leftarrow}(n/\delta_n)$. The sequence $(\delta_n)_{n\geq 1}$ itself appears as the sequence of location parameters common to both the off-line and on-line interview counting r.v.'s in the convergence result of Theorem 2.2; and the sequence $(n/\delta_n)_{n\geq 1}$ appears in the scalings of the Brownian bridges in the convergence analysis (see Lemmas 3.2 and 3.3 and the proof of Theorem 2.2). The sequence $(\varepsilon_n)_{n\geq 1}$ is the salary threshold sequence for the first-stage acceptance procedure in the on-line policies $(\pi_n)_{n\geq 1}$. It is useful to identify one additional auxiliary sequence of positive constants. For $F(x)$ in Case I for minima, define $(\gamma_n)_{n\geq 1}$ by

$$\gamma_n := F^{\leftarrow}(n/\delta_n)/c(n/\delta_n) \text{ for } n \geq 1, \tag{8}$$

and note that $\gamma_n \to \infty$ as $n \to \infty$. This sequence $(\gamma_n)_{n\geq 1}$ is the key sequence in describing differences in convergence rates and scalings between distribution functions in Case I for minima and distribution functions in Case III for minima (see (9)-(13) and the statement of Theorem 2.2).

In addition, throughout Sections 2 and 3, the sequence $(\tilde{c}_n)_{n\geq 1}$ of the first-stage budget constraint parameters in the on-line policies $(\pi_n)_{n\geq 1}$ is assumed to satisfy $0 < \tilde{c}_n < c_n$ and $\lim_{n\to\infty} \tilde{c}_n/c_n = 1$. Some immediate relationships between these parameters are given in the following lemma.

Lemma 2.1 *Given the sequences $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$ and auxiliary sequences $(\delta_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$, and let $\tau_n := F(\varepsilon_n)$ for $n \geq 1$. If $F(x)$ is in Case I for minima, then*

$$\begin{aligned} &\varepsilon_n \approx c_n/n, \delta_n \approx n/\tau_n, \lim_{n\to\infty}(c_n - \tilde{c}_n)/(n\varepsilon_n) = 0 \\ &\lim_{n\to\infty} \gamma_n((F^{\leftarrow}(\tau_n)/H(\tau_n)) - 1) = 1 \text{ and } \lim_{n\to\infty} \gamma_n F((c_n - \tilde{c}_n)/n)/\tau_n = 0, \end{aligned} \tag{9}$$

and if $F(x)$ is in Case III for minima with index $\alpha > 0$, and $a = -1/\alpha$, then

$$\varepsilon_n \approx (1-a)c_n/n, \delta_n \approx n/\tau_n, \lim_{n\to\infty}(c_n - \tilde{c}_n)/(n\varepsilon_n) = 0 \text{ and } \lim_{n\to\infty} F((c_n - \tilde{c}_n)/n)/\tau_n = 0. \tag{10}$$

Proof. These conclusions are straightforward applications of Karamata's Theorem and Representation for functions slowly varying at zero (see the Appendix), except for the result that $\lim_{n\to\infty} \gamma_n F((c_n - \tilde{c}_n)/n)/F(\varepsilon_n) = 0$ for $F(x)$ in Case I for minima. To see this last result, one may argue as follows. Let $g(t) := \int_0^t F(x)dx/F(t)$ for $t > 0$, and $x_n := (\varepsilon_n - ((c_n - \tilde{c}_n)/n))/g(\varepsilon_n)$ for $n \geq 1$, and observe that $x_n \approx \gamma_n$. Let $0 < \eta < 1/2$, and obtain that there is a positive constant K_η such that, for all n large,

$$\gamma_n \frac{F((c_n - \tilde{c}_n)/n)}{\tau_n} \approx x_n \frac{F(\varepsilon_n - x_n g(\varepsilon_n))}{F(\varepsilon_n)} \approx \bar{x}_n \frac{F(\varepsilon_n - \bar{x}_n \tilde{g}(\varepsilon_n))}{F(\varepsilon_n)} \leq K_\eta (\bar{x}_n)^{1-(1/(2\eta))} \to 0,$$

where $\tilde{g}(u)$ is an auxiliary function for the representation of $F(x)$ as given in the appendix, and $\bar{x}_n := x_n g(\varepsilon_n)/\tilde{g}(\varepsilon_n)$ for $n \geq 1$, with $\bar{x}_n \approx x_n$. □

The following three conditions are assumed to hold in the main theorem of this section. Conditions (11) and (12) are conditions on the sequence of budget constraints $(c_n)_{n\geq 1}$, and (13) connects the two sequences $(\tilde{c}_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ of budget constraints:

$$\begin{aligned}
\lim_{n\to\infty} n^{1/2}\left(\frac{F(F^{\leftarrow}(n/\delta_n))}{n/\delta_n}-1\right) &= 0 \quad , && (11)\\
\lim_{n\to\infty} \gamma_n^2/n^{1/2} &= 0 \quad \text{if } F(x) \text{ is in Case I}, && (12)\\
\lim_{n\to\infty} \gamma_n n^{1/2}(1-(\tilde{c}_n/c_n)) &= 0 \quad \text{if } F(x) \text{ is Case I, and} && (13)\\
\lim_{n\to\infty} n^{1/2}(1-(\tilde{c}_n/c_n)) &= 0 \quad \text{if } F(x) \text{ is in Case III.}
\end{aligned}$$

Condition (11) ensures that the location parameters coincide for the convergence of the on-line and off-line interview counting r.v.'s in Theorem 2.2. If d.f. $F(x)$ is continuous, then condition (11) clearly holds. Since $F(F^{\leftarrow}(s))/s \to 1$ as $s \to 0$ for these distribution functions, condition (11) can be interpreted as a condition on the speed of convergence of $F(F^{\leftarrow}(s))/s \to 1$ as $s \to 0$. Condition (12) limits the discrepancy in asymptotic behavior between certain sums of r.v.'s with number of terms indexed by a certain r.v. or indexed by this r.v.'s expectation. The weaker condition $\lim_{n\to\infty} \gamma_n/n^{1/2} = 0$ and the condition (13) are required to ensure nondegeneracy and stability in the key convergence of $\eta_n/n \to 1$ as $n \to \infty$ (see the proof of Theorem 2.2 for $F(x)$ in Case I, and Lemma 3.4; for (13) see (17) and (27)).

Theorem 2.2 *Let F be in Case I or III for minima with $l_F = 0$, and given sequences $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$, with auxiliary sequences $(\delta_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$. Assume that conditions* (11)-(13) *hold. Then there exist positive constants $(\vartheta_n)_{n\geq 1}$ and $(\tilde{\vartheta}_n)_{n\geq 1}$ for which*

$$\left(\vartheta_n^{-1}(M^e(c_n)-\delta_n), \tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n)\right) \Rightarrow (\mathcal{M}^e, \mathcal{M}^\pi)$$

where $(\mathcal{M}^e, \mathcal{M}^\pi) = (Y_1, Y_2^+)$ and $\mathbf{Y} = (Y_1, Y_2)$ is $N(\mathbf{0}, \Sigma_{\mathbf{Y}})$-distributed.

For $F(x)$ in Case I, the constants $(\vartheta_n)_{n\geq 1}$ and $(\tilde{\vartheta}_n)_{n\geq 1}$ satisfy $\vartheta_n \approx \delta_n/n^{1/2}$ and $\tilde{\vartheta}_n \approx n^{1/2}/(\gamma_n F((c_n-\tilde{c}_n)/n))$ for $(\gamma_n)_{n\geq 1}$ in (8), and $\Sigma_{\mathbf{Y}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

For $F(x)$ in Case III with index $\alpha > 0$, and $a = -1/\alpha$, the constants $(\vartheta_n)_{n\geq 1}$ and $(\tilde{\vartheta}_n)_{n\geq 1}$ satisfy $\vartheta_n \approx \delta_n/n^{1/2}$ and $\tilde{\vartheta}_n \approx n^{1/2}/F((c_n-\tilde{c}_n)/n)$, and $\Sigma_{\mathbf{Y}} = \begin{pmatrix} \frac{2(1-a)}{1-2a} & \frac{-a}{1-2a} \\ \frac{-a}{1-2a} & \frac{a^2}{1-2a} \end{pmatrix}$.

Observe that $\vartheta_n, \tilde{\vartheta}_n \to \infty$ with $\vartheta_n/\tilde{\vartheta}_n \to 0$ as $n \to \infty$ (from (9) and (10)). The inequality $M^e(c_n) \leq M^{\pi_n}(c_n)$ carries over (with probability one) to the limit r.v.'s as $\mathcal{M}^\pi \geq 0$, since $\psi_n \vartheta_n^{-1}(M^e(c_n)-\delta_n) \leq \tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n)$ where $\psi_n = \vartheta_n/\tilde{\vartheta}_n$.

Example 2.3 For distribution function $F(x) = x^{-1/a}$, $0 \leq x \leq 1$, $F(x)$ is in Case III for minima, with $l_F = 0$ and with $-\infty < a < 0$. Thus, $n^{-a}\min_{1\leq i\leq n} X_i \Rightarrow$ an K_{III}^α-distributed r.v., where d.f. $K_{\text{III}}^\alpha(x)$ is given in the Appendix. For $0 < s < 1$, $F^{\leftarrow}(s) = s^{-a}$ and $H(s) = (-a+1)^{-1}s^{-a}$; and for $0 \leq x \leq (-a+1)^{-1}$, $H^{\leftarrow}(x) = ((-a+1)x)^{-1/a}$. For positive constants $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$ satisfying (13), it follows that $\delta_n = \left((1-a)c_n n^{a-1}\right)^{1/a}$ and

$\varepsilon_n = (1-a)(c_n/n)$. Let $\kappa_n = F(\varepsilon_n)/F((c_n - \tilde{c}_n)/n)$, so that $\tilde{\vartheta}_n \approx \kappa_n \vartheta_n$ with $\vartheta_n \approx \delta_n/n^{1/2}$. Theorem 2.2 implies

$$\left(n^{1/2}\left(\frac{M^e(c_n)}{\delta_n} - 1\right), \frac{n^{1/2}}{\kappa_n}\left(\frac{M^{\pi_n}(c_n)}{\delta_n} - 1\right)\right) \Rightarrow (\mathcal{M}^e, \mathcal{M}^\pi)$$

where $(\mathcal{M}^e, \mathcal{M}^\pi)$ is given in Theorem 2.2. Note that $\kappa_n \to \infty$ with $\kappa_n = (1-a)^{-1/a}\left(1 - \frac{\tilde{c}_n}{c_n}\right)^{1/a} = o(n^{-1/(2a)})$.

In particular, if $F(x)$ is the d.f. of a r.v. uniformly distributed on $(0,1)$, then $F(x)$ is in Case III for minima with $a = -1$, and $\delta_n = n^2/(2c_n)$ and $\kappa_n = 2\left(1 - \frac{\tilde{c}_n}{c_n}\right)^{-1}$. For example, if $c_n := n/\log n$ and $\tilde{c}_n := (n/\log n)(1 - n^{-1/2}(\log n)^{-1})$, then we have the simple calculation for any $\kappa > 0$,

$$\lim_{n\to\infty} P\left(\left(M^{\pi_n}\left(\frac{n}{\log n}\right) - \frac{n \log n}{2}\right) < \left(M^e\left(\frac{n}{\log n}\right) - \frac{n\log n}{2}\right)^2 + \kappa n(\log n)^2\right)$$

$$= P(Y_2 < \tfrac{1}{4}Y_1^2 + \kappa)$$

where (Y_1, Y_2) is multivariate normal distributed with mean $(0,0)$ and covariance matrix $\frac{1}{3}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$.

Example 2.4 For distribution function $F(x) = e^{-1/x}$ for $x > 0$, $F(x)$ is in Case I for minima with $l_F = 0$. Thus, $(\log n)^2\left(\min_{1\le i\le n} X_i - (\log n)^{-1}\right) \Rightarrow$ an K_{I}-distributed r.v., where $K_{\mathrm{I}}(x)$ is given in the Appendix. For this d.f., the auxiliary function $g(t) := \int_0^t F(x)dx/F(t)$ satisfies $g(t) \approx t^2$ as $t \to 0$; $F^{\leftarrow}(s) = -(\log s)^{-1}$ for $0 < s < 1$; and $H(s) \approx -(\log s)^{-1}$ and $c(s) \approx (\log s)^{-2}$ as $s \to 0$. Let $(\omega_n)_{n\ge1}$ and $(\rho_n)_{n\ge1}$ be the following two basic data sequences of positive constants. Sequence $(\omega_n)_{n\ge1}$ is given satisfying $\lim_{n\to\infty}\omega_n = \infty$ and $\lim_{n\to\infty}\omega_n^2/n^{1/2} = 0$; and define $(c_n)_{n\ge1}$ by $c_n = nH(e^{-\omega_n})$ for $n \ge 1$. Then the sequences $(\delta_n)_{n\ge1}$, $(\varepsilon_n)_{n\ge1}$ and $(\gamma_n)_{n\ge1}$ satisfy $\delta_n = ne^{\omega_n}$, $\varepsilon_n = \omega_n^{-1}$ and $\gamma_n \approx \omega_n$. Sequence $(\rho_n)_{n\ge1}$ is given satisfying $0 < \rho_n < 1$, $\lim_{n\to\infty}\rho_n = 1$ and $\lim_{n\to\infty} n^{1/2}\omega_n(1-\rho_n) = 0$; and define $(\tilde{c}_n)_{n\ge1}$ by $\tilde{c}_n = \rho_n c_n$ for $n \ge 1$. Note that conditions (11)-(13) hold. Theorem 2.2 implies

$$\left(n^{1/2}\left(\frac{M^e(c_n)}{\delta_n} - 1\right), \frac{n^{1/2}}{\kappa_n}\left(\frac{M^{\pi_n}(c_n)}{\delta_n} - 1\right)\right) \Rightarrow (\mathcal{M}^e, \mathcal{M}^\pi)$$

where $(\mathcal{M}^e, \mathcal{M}^\pi)$ are given in Theorem 2.2. Here $\kappa_n \to \infty$ with $\kappa_n = \exp(((c_n/n)(1-\rho_n))^{-1} - \omega_n - \log \omega_n) = \exp((\omega_n/(1-\rho_n))(1 + o((\log \omega_n)/\omega_n)))$ for $n \ge 1$.

Example 2.5 This example shows the necessity of condition (11) for Theorem 2.2 and Lemma 3.4 (*iii*). Let $F(x)$ be given by $F(x) = k^{-3}$ if $(k+1)^{-1} \le x < k^{-1}$, for $k = 1, 2, \cdots$; so $F(x)$ is in Case III for minima with $l_F = 0$, and with $\alpha = 3$ and $a = -1/3$, and $n^{1/3}\min_{1\le i\le n} X_i \Rightarrow K_{\mathrm{III}}^3$. Its inverse function is given by $F^{\leftarrow}(s) = (k+1)^{-1}$ if $(k+1)^{-3} < s \le k^{-3}$, $k = 1, 2, \cdots$. Let $k_n := \lfloor \log n \rfloor$ for $n \ge 1$ and $(\psi_n)_{n\ge1}$ be constants satisfying $0 < \psi_n < 1$ and $\psi_n \uparrow 1$ as $n \to \infty$; and let $(c_n)_{n\ge1}$ be the positive constants satisfying

$$
\begin{aligned}
c_n/n &= H((k_n+\psi_n)^3) \\
&= (k_n+\psi_n)^3\left(\sum_{l=k_n+1}^{\infty} l^{-1}((l-1)^{-3}-l^{-3})-(k_n+1)^{-1}(k_n^{-3}-(k_n+\psi_n)^{-3})\right) \\
&= (k_n+\psi_n)^3\left((k_n+\psi_n)^{-3}(k_n+1)^{-1}-\sum_{l=k_n+1}^{\infty} l^{-4}(l+1)^{-1}\right).
\end{aligned}
$$

Thus, $c_n \approx (3/4)(n/\log n)$ as $n \to \infty$. These constants $(c_n)_{n\geq 1}$ are chosen so that constants $(\delta_n)_{n\geq 1}$ of (7) are given by $\delta_n = n(k_n+\psi_n)^3$ for $n \geq 1$, so that $\delta_n \approx n(\log n)^3$ as $n \to \infty$. Then constants $(\varepsilon_n)_{n\geq 1}$ of (7) are given by $\varepsilon_n = (\lceil \log n \rceil)^{-1}$, and $n^{1/2}((F(F^{\leftarrow}(n/\delta_n))/(n/\delta_n))-1) \approx 3n^{1/2}/\log n \to \infty$ as $n \to \infty$. Let $(\tilde{c}_n)_{n\geq 1}$ be any positive constants satisfying (13). Using the methods of proof of Lemma 3.4 and Theorem 2.2, one can show that $\eta_n/n \to 1$ and $n^{-1/2}(n-\eta_n) \to \infty$ in probability as $n \to \infty$; and $\vartheta_n^{-1}(M^e(c_n)-\delta_n) \Rightarrow \mathcal{M}^e$, a $N(0, 8/5)$-distributed r.v., but $\tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n) \to \infty$ in probability as $n \to \infty$. Here, $\vartheta_n \approx n^{1/2}(\log n)^3$ and $\tilde{\vartheta}_n \approx n^{1/2}(\log n)^3((4/3)(1-(\tilde{c}_n/c_n))^{-1})^3$ as $n \to \infty$.

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.6 *Let $F(x)$ be in Case I or III for minima with $l_F = 0$, and given sequences $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$ with auxiliary sequences $(\delta_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ as in Theorem 2.2. Then*
(i) the sequence of two-stage threshold policies $(\pi_n)_{n\geq 1}$ is a consistent approximator of the off-line smallest interview-count strategy τ^e;
(ii) if $\lim_{n\to\infty} \tilde{\vartheta}_n/\delta_n = 0$, then $M^{\pi_n}(c_n)/M^e(c_n) \to 1$ in probability as $n \to \infty$ (for $F(x)$ in Case III, this condition holds if $a < -1$ and $\lim_{n\to\infty} n^{-a/2}(1-(\tilde{c}_n/c_n)) = \infty$); and
(iii) if $\mathcal{M}^e$ and $\mathcal{M}^\pi$ are given as in Theorem 2.2, then

$$
\frac{\gamma_n F((c_n-\tilde{c}_n)/n)}{F(\varepsilon_n)}\left(\frac{M^{\pi_n}(c_n)-\delta_n}{M^e(c_n)-\delta_n}\right) \Rightarrow \frac{\mathcal{M}^\pi}{\mathcal{M}^e} \text{ if } F(x) \text{ is in Case I, and}
$$

$$
\frac{F((c_n-\tilde{c}_n)/n)}{F(\varepsilon_n)}\left(\frac{M^{\pi_n}(c_n)-\delta_n}{M^e(c_n)-\delta_n}\right) \Rightarrow \frac{\mathcal{M}^\pi}{\mathcal{M}^e} \text{ if } F(x) \text{ is in Case III.}
$$

3 Proofs of main results for medium-sized budget constraints

For the proof of Theorem 2.2, it is assumed that the underlying probability space is that of Csörgő et al. [6] carrying an infinite sequence $U_1, U_2, \ldots$ of i.i.d. uniform $(0,1)$ r.v.'s and a sequence $U_n(s)$, $0 \leq s \leq 1$, $n = 1, 2, \ldots$, of Brownian bridges for which the following Brownian bridge approximation to the uniform empirical process holds:

$$
\sup_{1/n \leq s \leq 1-1/n} n^{\nu}\frac{|\alpha_n(s)-U_n(s)|}{(s(1-s))^{(1/2)-\nu}} = O_P(1) \text{ as } n \to \infty, \tag{14}
$$

where $\alpha_n(s) = n^{1/2}(G_n(s)-s)$, $G_n(s) = n^{-1}\sum_{i=1}^{n} I(U_i \leq s)$, and ν is any fixed number such that $0 \leq \nu < 1/4$. This can be assumed without loss of generality. On this space we use $X_i = F^{\leftarrow}(U_i)$, for $i = 1, 2, \ldots$, and have the following representations.

Lemma 3.1 *Let $F(x)$ be in Case I or III for minima with $l_F = 0$, and given sequence $(c_n)_{n\geq 1}$ with auxiliary sequence $(\delta_n)_{n\geq 1}$. Let $(j_n)_{n\geq 1}$ and $(m_n)_{n\geq 1}$ be any sequences of positive integers such that $j_n \approx \delta_n$ and $m_n \approx \delta_n$. Then*

$$\frac{\sum_{i=1}^{n} X_i^{(j_n)} - j_n \int_0^{n/j_n} F^{\leftarrow}(s)ds}{j_n^{1/2} A^e(n/j_n)} = -\frac{\int_{1/\delta_n}^{n/\delta_n} U_{m_n}(s)dF^{\leftarrow}(s)}{A^e(n/\delta_n)} + o_P(1)$$

where for $0 < s < 1$, $A^e(s) = \begin{cases} s^{1/2}c(s) & \text{in Case I} \\ s^{1/2}F^{\leftarrow}(s) & \text{in Case III} \end{cases}$.

Proof. One uses standard arguments of Csörgő and Mason [8], Csörgő, Haeusler and Mason [9] and Lo [11] to justify the following equalities:

$$\begin{aligned}
&\left(j_n^{1/2}A^e(n/j_n)\right)^{-1}\left(\sum_{i=1}^{n} X_i^{(j_n)} - j_n \int_0^{n/j_n} F^{\leftarrow}(s)ds\right) \\
&= (A^e(n/j_n))^{-1}\left(\int_{1/j_n}^{n/j_n} \alpha_{j_n}(s)dF^{\leftarrow}(s)\right) + o_P(1) \\
&= (A^e(n/\delta_n))^{-1}\left(\int_{1/\delta_n}^{n/\delta_n} \alpha_{j_n}(s)dF^{\leftarrow}(s)\right) + o_P(1) \\
&= (A^e(n/\delta_n))^{-1}\left(\int_{1/\delta_n}^{n/\delta_n} \alpha_{m_n}(s)dF^{\leftarrow}(s)\right) + o_P(1). \quad \square
\end{aligned}$$

Lemma 3.2 *Let $F(x)$ be in Case I or III for minima with $l_F = 0$, and given sequence $(c_n)_{n\geq 1}$ with auxiliary sequences $(\delta_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$. Let $(m_n)_{n\geq 1}$ be any sequence of positive integers such that $m_n \approx \delta_n$, and let $\tau_n := F(\varepsilon_n)$ for $n \geq 1$. Then*

$$\begin{aligned}
&(m_n\tau_n)^{-1/2}\left(\sum_{i=1}^{m_n} I(X_i \leq \varepsilon_n) - m_n\tau_n\right) \\
&= \tau_n^{-1/2}U_{m_n}(\tau_n) + o_P(1) = (n/\delta_n)^{-1/2}U_{m_n}(n/\delta_n) + o_P(1)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\sum_{i=1}^{m_n} X_i I(X_i \leq \varepsilon_n) - m_n \int_0^{\tau_n} F^{\leftarrow}(s)ds}{(m_n\tau_n)^{1/2}F^{\leftarrow}(\tau_n)} \\
&= \begin{cases} \tau_n^{-1/2}U_{m_n}(\tau_n) + o_P(1) & \text{in Case I} \\ \tau_n^{-1/2}U_{m_n}(\tau_n) - \dfrac{\int_{\tau_n/n}^{\tau_n} U_{m_n}(s)dF^{\leftarrow}(s)}{\tau_n^{1/2}F^{\leftarrow}(\tau_n)} + o_P(1) & \text{in Case III} \end{cases}.
\end{aligned}$$

If, in addition, condition (11) holds, then these conclusions hold with τ_n replaced by n/δ_n.

Proof. The proof is straightforward, given the underlying probability structures associated with (14). For $F(x)$ in Case I, verification of the second equality uses results from the Theorem of Lo [11] and $\lim_{s\to 0} c(s)/F^{\leftarrow}(s) = 0$. $\square$

Asymptotic behavior of the on-line policy's first stage 'applicant-selection' r.v.'s $(k_{i,n})_{i\geq 1}$ and $(W_{i,n})_{i\geq 1}$, and first-stage-termination r.v. η_n, as $n \to \infty$, is given in the following lemmas.

Lemma 3.3 *Let $F(x)$ be in Case I or III for minima with $l_F = 0$, and given sequence $(c_n)_{n\geq 1}$ with auxiliary sequences $(\delta_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$, and let $\tau_n := F(\varepsilon_n)$ for $n \geq 1$. Let $(k_{i,n})_{i\geq 1}$ be the i.i.d. r.v.'s defined in (2), for each $n = 1, 2, \ldots$, and denote $m_n = E\left(\sum_{i=1}^n k_{i,n}\right)$ and $s_n^2 = \mathrm{Var}\left(\sum_{i=1}^n k_{i,n}\right)$ for $n \geq 1$. Then for each real number r,*

$$\begin{aligned}\left\{s_n^{-1}\left(\sum_{i=1}^n k_{i,n} - m_n\right) \leq r\right\} &= \left\{-(m_n\tau_n)^{-1/2}\sum_{i=1}^{m_n}(I(X_i \leq \varepsilon_n) - \tau_n) + o_P(1) \leq r\right\} \qquad (15)\\ &= \left\{-(n/\delta_n)^{-1/2}U_{m_n}(n/\delta_n) + o_P(1) \leq r\right\}\end{aligned}$$

and hence $\left\{\left(\sum_{i=1}^n k_{i,n} - (n/\tau_n)\right)/(n^{1/2}/\tau_n)\right\}_{n\geq 1}$ converges in distribution to a $N(0,1)$-distributed r.v. as $n \to \infty$. If, in addition, condition (11) holds, then the conclusions hold with τ_n replaced by n/δ_n.

Proof. We prove (15). Let $\mu_n = \lfloor m_n + rs_n \rfloor$ for $n \geq 1$, and note that $(n - \mu_n\tau_n)/(m_n\tau_n(1-\tau_n))^{1/2} = -r + o(1)$. ¿From the definition of $(k_{i,n})_{i\geq 1}$ and Lemma 3.2, obtain that

$$\begin{aligned}\left\{s_n^{-1}\left(\sum_{i=1}^n k_{i,n} - m_n\right) \leq r\right\} &= \left\{\sum_{i=1}^n k_{i,n} \leq \mu_n\right\}\\ &= \left\{\sum_{i=1}^{\mu_n} I(X_i \leq \varepsilon_n) \geq n\right\}\\ &= \left\{m_n^{-1/2}\sum_{i=1}^{\mu_n}\left(\frac{I(X_i \leq \varepsilon_n) - \tau_n}{(\tau_n(1-\tau_n))^{1/2}}\right) \geq \frac{n - \mu_n\tau_n}{(m_n\tau_n(1-\tau_n))^{1/2}}\right\}\\ &= \left\{(m_n\tau_n)^{-1/2}\sum_{i=1}^{m_n}(I(X_i \leq \varepsilon_n) - \tau_n) + o_P(1) \geq -r\right\}\\ &= \left\{-(n/\delta_n)^{-1/2}U_{m_n}(n/\delta_n) + o_P(1) \leq r\right\} \quad \square\end{aligned}$$

Observe that in (15) the $o_P(1)$ terms depend on r; and we do *not* claim that $s_n^{-1}(\sum_{i=1}^n k_{i,n} - m_n) = -(n/\delta_n)^{-1/2}U_{m_n}(n/\delta_n) + o_P(1)$ for some $o_P(1)$ term independent of problem parameters. This complicates the reasoning in the proof of Theorem 2.2 somewhat.

Lemma 3.4 *Let $F(x)$ be in Case I or III for minima with $l_F = 0$, and given sequences $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$ with auxiliary sequences $(\delta_n)_{n\geq 1}$, $(\varepsilon_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$. Then*
(i) $\eta_n/n \to 1$ in probability as $n \to \infty$ (where η_n is defined in (3));
(ii) $n^{-1/2}\sum_{i=1}^n(W_{i,n} - EW_{i,n})/(\mathrm{Var}\, W_{i,n})^{1/2} \Rightarrow N(0,1)$-distributed r.v., for $F(x)$ in Case III and for $F(x)$ in Case I satisfying $\lim_{n\to\infty}\gamma_n/n^{1/2} = 0$;
(iii) if (11) and (13) hold, and $\lim_{n\to\infty}\gamma_n/n^{1/2} = 0$, then

$$\begin{aligned}&(n - \eta_n)/(n^{1/2}/\gamma_n) \Rightarrow Z_1^+ && \text{if} \quad F(x) \text{ is in Case I, and}\\ &(n - \eta_n)/n^{1/2} \Rightarrow (-a/(1-2a)^{1/2})Z_2^+ && \text{if} \quad F(x) \text{ is in Case III,}\end{aligned}$$

where Z_1 and Z_2 are $N(0,1)$-distributed r.v.'s.

Proof. Let $\tau_n = F(\varepsilon_n)$ for $n \geq 1$ and consider the r.v.'s $(W_{i,n})_{i\geq 1}$ of (2) for $n \geq 1$. First, note the characterizations and behavior of the following constants associated with these r.v.'s:

$$EW_{i,n} = H(\tau_n) \text{ and } \operatorname{Var} W_{i,n} = \tau_n^{-1} \int_0^{\tau_n} (F^{\leftarrow}(s))^2 ds - (H(\tau_n))^2.$$

For $0 < \nu < 1$, let $\rho_n = \lfloor n(1-\nu) \rfloor$ for $n \geq 1$. If $F(x)$ is in Case I, then $\operatorname{Var} W_{i,n} \approx (c(\tau_n))^2$ and for $n \to \infty$

$$\frac{\tilde{c}_n - \rho_n EW_{i,n}}{(\rho_n \operatorname{Var} W_{i,n})^{1/2}} \approx \frac{c_n\left(\frac{\tilde{c}_n}{c_n} - \frac{\rho_n}{n}\frac{H(\tau_n)}{H(n/\delta_n)}\right)}{\rho_n^{1/2}\frac{c(n/\delta_n)}{F^{\leftarrow}(n/\delta_n)}\frac{F^{\leftarrow}(n/\delta_n)}{H(n/\delta_n)}\frac{c_n}{n}} \approx \frac{n^{1/2}F^{\leftarrow}(n/\delta_n)\nu}{(1-\nu)^{1/2}c(n/\delta_n)} \to \infty; \tag{16}$$

and if $F(x)$ is in Case III and $\tilde{K}_a := -a/((1-2a)^{1/2}(1-a))$, then $\operatorname{Var} W_{i,n} \approx \tilde{K}_a^2(F^{\leftarrow}(\tau_n))^2$ and for $n \to \infty$

$$\frac{\tilde{c}_n - \rho_n EW_{i,n}}{(\rho_n \operatorname{Var} W_{i,n})^{1/2}} \approx \frac{c_n\left(\frac{\tilde{c}_n}{c_n} - \frac{\rho_n}{n}\frac{H(\tau_n)}{H(n/\delta_n)}\right)}{\rho_n^{1/2}\tilde{K}_a\frac{F^{\leftarrow}(\tau_n)}{H(\tau_n)}\frac{H(\tau_n)}{H(n/\delta_n)}\frac{c_n}{n}} \approx \frac{n^{1/2}\nu(1-2a)^{1/2}}{(1-\nu)^{1/2}(-a)} \to \infty.$$

Thus, it follows from the definition of η_n and from Markov's inequality that

$$\begin{aligned} P(|(\eta_n/n) - 1| > \nu) &= P(\eta_n \leq \rho_n) = P\left(\sum_{i=1}^{\rho_n} W_{i,n} > \tilde{c}_n\right) \\ &= P\left(\frac{\sum_{i=1}^{\rho_n}(W_{i,n} - EW_{i,n})}{(\rho_n \operatorname{Var} W_{i,n})^{1/2}} > \frac{\tilde{c}_n - \rho_n EW_{i,n}}{(\rho_n \operatorname{Var} W_{i,n})^{1/2}}\right) = o(1), \end{aligned}$$

and so $\eta_n/n \to 1$ in probability as $n \to \infty$.

The convergence in distribution of the normalized sums $n^{-1/2}\sum_{i=1}^n(W_{i,n} - EW_{i,n})/(\operatorname{Var} W_{i,n})^{1/2}$ to a $N(0,1)$-distributed r.v. follows, for example, from the Central Limit Theorem for arrays given in Chung [4, page 200], under the additional hypothesis $\lim_{n\to\infty}\gamma_n/n^{1/2} = 0$ for $F(x)$ in Case I for minima.

For the proof of (iii) under (11) and (13) and $\lim_{n\to\infty}\gamma_n/n^{1/2} = 0$, observe first that $H(\tau_n) = H(n/\delta_n) + (1 - ((n/\delta_n)/\tau_n))(F^{\leftarrow}(n/\delta_n) - H(n/\delta_n))$. Use this to show that for $r \geq 0$, if $F(x)$ is in Case I and $l_n := \lfloor n - (n^{1/2}/\gamma_n)r \rfloor$, then

$$\begin{aligned} \frac{\tilde{c}_n - l_n EW_{i,n}}{l_n^{1/2}(\operatorname{Var} W_{i,n})^{1/2}} &= \gamma_n n^{1/2}\left(\frac{\tilde{c}_n}{c_n} - 1\right)(1 + o(1)) - n^{1/2}\left(1 - \frac{n/\delta_n}{\tau_n}\right)(1 + o(1)) \\ &\quad + r(1 + o(1)) + o(1) \end{aligned} \tag{17}$$

and

$$\{(n - \eta_n)/(n^{1/2}/\gamma_n) < r\} = \left\{\sum_{i=1}^{l_n} W_{i,n} \leq \tilde{c}_n\right\};$$

and if $F(x)$ is in Case III and $l_n := \lfloor n - n^{1/2}r \rfloor$, then

$$\begin{aligned} \frac{\tilde{c}_n - l_n EW_{i,n}}{l_n^{1/2}(\operatorname{Var} W_{i,n})^{1/2}} &= n^{1/2}\left(\frac{\tilde{c}_n}{c_n} - 1\right)\left(\frac{1}{1-a} + o(1)\right) \\ &\quad - n^{1/2}\left(1 - \frac{n/\delta_n}{\tau_n}\right)\left(\frac{-a}{1-a} + o(1)\right) + r\left((\tilde{K}_a(1-a))^{-1} + o(1)\right) + o(1) \end{aligned} \tag{18}$$

and

$$\{n^{-1/2}(n-\eta_n)<r\}=\left\{\sum_{i=1}^{l_n}W_{i,n}\le\tilde{c}_n\right\}.$$

The conclusion to part (iii) now follows from (11), (13), and $\lim_{n\to\infty}\gamma_n/n^{1/2}=0$ and part (ii). $\square$

The norming constants can be characterized further in Lemma 3.4 (ii), in terms of the given sequences of positive constants, if $F(x)$ also satisfies condition (11). In this case

$$\left(\sum_{i=1}^{n}W_{i,n}-c_n\right)/((n^{1/2}/\gamma_n)\varepsilon_n)\quad\Rightarrow\quad N(0,1)-\text{distributed r.v. if } F(x) \text{ is in Case I} \tag{19}$$

and

$$\left(\sum_{i=1}^{n}W_{i,n}-c_n\right)/(n^{1/2}\tilde{K}_a\varepsilon_n)\quad\Rightarrow\quad N(0,1)-\text{distributed r.v. if } F(x) \text{ is in Case III}$$

where $\tilde{K}_a=-a/((1-2a)^{1/2}(1-a))$. The results in Lemmas 3.1-3.4 are used in the proof of Theorem 2.2. In addition, to obtain the desired joint convergence in Theorem 2.2, strong convergence improvement of Lemma 3.4 (iii) is needed; this is given within the following proof.

Proof of Theorem 2.2. The theorem is proved first for d.f. $F(x)$ in Case III for minima with $l_F=0$.

First, from definition (1) for $M^e(c_n)$, Lemma 3.1, and Karamata's Theorem, we obtain, for each real number μ, and $j_n=\lfloor\delta_n+\vartheta_n\mu\rfloor$, that

$$\begin{aligned}\{\vartheta_n^{-1}(M^e(c_n)-\delta_n)\le\mu\}&=\left\{\sum_{i=1}^{n}X_i^{(j_n)}\le c_n\right\}\\ &=\left\{\frac{\sum_{i=1}^{n}X_i^{(j_n)}-j_n\int_0^{j_n/n}F^{\leftarrow}(s)ds}{j_n^{1/2}A^e(n/j_n)}\le\frac{nH(n/\delta_n)-nH(n/j_n)}{j_n^{1/2}A^e(n/j_n)}\right\}\\ &=\left\{-\left(\frac{1-a}{-a}\right)\frac{\int_{1/\delta_n}^{n/\delta_n}U_{m_n}(s)dF^{\leftarrow}(s)}{A^e(n/\delta_n)}+o_P(1)\le\mu\right\}\end{aligned}\tag{20}$$

where $A^e(s)=s^{1/2}F^{\leftarrow}(s)$ and $m_n=n/F(\varepsilon_n)$ for $n\ge1$. Next, from representation (5) for $M^{\pi_n}(c_n)$, and using (10) and assumption (11), we have

$$\begin{aligned}\tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n)&=\tilde{\vartheta}_n^{-1}\sum_{i=1}^{\eta_n\wedge n}(k_i-1/F(\varepsilon_n))+\tilde{\vartheta}_n^{-1}\sum_{i=1}^{n+1-\eta_n}(k_i'-1/F((c_n-\tilde{c}_n)/n))\\ &\quad+\tilde{\vartheta}_n^{-1}(n-\eta_n)(1/F((c_n-\tilde{c}_n)/n))-(1/F(\varepsilon_n)))+o(1)\\ &=:\ \mathrm{I}_n+\mathrm{II}_n+\mathrm{III}_n+o(1).\end{aligned}\tag{21}$$

(Here the full extent of (11) is not used, but only that $n^{1/2}((F(F^{\leftarrow}(n/\delta_n))/(n/\delta_n))-1)=O(1)$, in order to replace the location parameter $\delta_n=n/H^{\leftarrow}(c_n/n)$ by $n/F(\varepsilon_n)$.)

It follows from Lemmas 3.3 and 3.4, and an argument analogous to that in the proof of the Doeblin-Anscombe Theorem ([3, page 317]), together with (10), that $\mathrm{I}_n = o_P(1)$. If we denote $p_n = F((c_n - \tilde{c}_n)/n)$, then $\mathrm{II}_n = (1-p_n)^{1/2} n^{-1/2} \sum_{i=1}^{n+1-\eta_n} \hat{k}'_i$, where $\hat{k}'_i = (k'_i - p_n^{-1})/((1-p_n)^{1/2}/p_n)$. And we have $n^{-1/2}\sum_{i=1}^{n+1-\eta_n} \hat{k}'_i = o_P(1)$, since, for any $0 < \nu, \gamma < 1$,

$$
\begin{aligned}
P\left(\left|\sum_{i=1}^{n+1-\eta_n} \hat{k}'_i\right| > n^{1/2}\nu\right) &\leq P\left(\left|\sum_{i=1}^{n+1-\eta_n} \hat{k}'_i\right| > n^{1/2}\nu,\ |n-\eta_n| \leq \gamma n\right) + o(1) \\
&\leq P\left(\max_{1\leq j \leq \lceil \gamma n\rceil}\left|\sum_{i=1}^{j} \hat{k}'_i\right| > n^{1/2}\nu\right) + o(1) \leq \nu^{-2} n^{-1} E\left(\sum_{i=1}^{\lceil \gamma n\rceil} \hat{k}'_i\right)^2 + o(1) \leq 2\nu^{-2}\gamma + o(1)
\end{aligned}
$$

for all n large, where the first inequality follows from Lemma 3.4. Thus, $\mathrm{II}_n = o_P(1)$ also, and we may conclude that

$$
\tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n) - \delta_n) = n^{-1/2}(n-\eta_n) + o_P(1). \tag{22}
$$

Thus, the asymptotic analysis of $(M^e(c_n), M^{\pi_n}(c_n))$ is facilitated through further analysis of η_n.

Now let $K_n := k_{1,n} + \cdots + k_{n,n}$, using r.v.'s $(k_{i,n})_{i\geq 1}$ of (2), and $m_n = EK_n = n/F(\varepsilon_n)$, for $n \geq 1$; further, denote $\phi_n = E(X_i I(X_i \leq \varepsilon_n))$ for $n \geq 1$. We obtain for $\tau \geq 0$ and $l_n = \lfloor n - n^{1/2}(\tau - o_P(1))\rfloor$ that

$$
\begin{aligned}
\{n^{-1/2}(n-\eta_n) + o_P(1) < \tau\} &= \left\{\sum_{i=1}^{l_n} W_{i,n} \leq \tilde{c}_n\right\} \\
&= \left\{\frac{\sum_{i=1}^{l_n}(W_{i,n} - EW_{i,n})}{n^{1/2}F^{\leftarrow}(n/\delta_n)} \leq \frac{\tilde{c}_n - l_n EW_{i,n}}{n^{1/2}F^{\leftarrow}(n/\delta_n)}\right\} \\
&= \left\{\frac{\sum_{i=1}^{n}(W_{i,n} - EW_{i,n})}{n^{1/2}F^{\leftarrow}(n/\delta_n)} + o_P(1) \leq \frac{\tau}{1-a}\right\} \\
&= \left\{\frac{\sum_{i=1}^{K_n} X_i I(X_i \leq \varepsilon_n) - nEW_{i,n}}{n^{1/2}F^{\leftarrow}(n/\delta_n)} + o_P(1) \leq \frac{\tau}{1-a}\right\} \\
&= \left\{\frac{\sum_{i=1}^{K_n}(X_i I(X_i \leq \varepsilon_n) - \phi_n)}{n^{1/2}F^{\leftarrow}(n/\delta_n)} + \frac{\phi_n(K_n - m_n)}{n^{1/2}F^{\leftarrow}(n/\delta_n)} + o_P(1) \leq \frac{\tau}{1-a}\right\} \\
&= \left\{(1-a)\frac{\sum_{i=1}^{K_n}(X_i I(X_i \leq \varepsilon_n) - \phi_n)}{n^{1/2}F^{\leftarrow}(n/\delta_n)} + \frac{K_n - m_n}{(\operatorname{Var} K_n)^{1/2}} + o_P(1) \leq \tau\right\}
\end{aligned}
\tag{23}
$$

Here, the third equality uses (18), Lemma 3.4 (*ii*) and, e.g., a Doeblin-Anscombe type reasoning as above; the fourth equality uses the definitions of $(k_{i,n})_{i\geq 1}$ and $(W_{i,n})_{i\geq 1}$; the fifth equality uses $nEW_{i,n} = m_n\phi_n$; and the sixth equality uses Lemma 3.3 and $\operatorname{Var} K_n \approx n^{1/2}/F(\varepsilon_n) \approx (1-a)n^{1/2}\varepsilon_n/\phi_n$.

To obtain the desired joint convergence, first use (20) with Lemmas 3.1-3.3 to obtain that for any real numbers μ, z_1 and z_2,

$$
\begin{aligned}
&P\left(\frac{M^e(c_n)-\delta_n}{\vartheta_n}\le\mu,\ \frac{K_n-m_n}{(\operatorname{Var}K_n)^{1/2}}\le z_1,\ (1-a)\frac{\sum_{i=1}^{m_n}(X_iI(X_i\le\varepsilon_n)-\phi_n)}{n^{1/2}F^{\leftarrow}(n/\delta_n)}\le z_2\right)\\
&= P\left(\sum_{i=1}^{n}X_i^{(j_n)}\le c_n,\ -\frac{\sum_{i=1}^{m_n}(I(X_i\le\varepsilon_n)-(n/\delta_n))}{n^{1/2}}\le z_1,\right.\\
&\qquad\left.(1-a)\frac{\sum_{i=1}^{m_n}(X_iI(X_i\le\varepsilon_n)-\phi_n)}{n^{1/2}F^{\leftarrow}(n/\delta_n)}\le z_2\right)+o(1)\\
&= P\left(-\left(\frac{1-a}{-a}\right)\frac{\int_{1/\delta_n}^{n/\delta_n}U_{m_n}(s)dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2}F^{\leftarrow}(n/\delta_n)}\le\mu,\ -\frac{U_{m_n}(n/\delta_n)}{(n/\delta_n)^{1/2}}\le z_1,\right. \qquad (24)\\
&\qquad\left.(1-a)\left(\frac{U_{m_n}(n/\delta_n)}{(n/\delta_n)^{1/2}}-\frac{\int_{1/\delta_n}^{n/\delta_n}U_{m_n}(s)dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2}F^{\leftarrow}(n/\delta_n)}\right)\le z_2\right)+o(1)\\
&= P(W_1\le\mu,\ W_2\le z_1,\ W_3\le z_2)+o(1),
\end{aligned}
$$

where (W_1, W_2, W_3) is multivariate normal distributed. Hence, we have from (20)-(23), and a slight variation of the Doeblin-Anscombe type reasoning based on the weak convergence conclusion of Lemma 3.3 that, for any numbers μ and η,

$$
\begin{aligned}
&P(\vartheta_n^{-1}(M^e(c_n)-\delta_n)\le\mu,\ \tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n)<\eta)\\
&\quad= P(\vartheta_n^{-1}(M^e(c_n)-\delta_n)\le\mu,\ n^{-1/2}(n-\eta_n)+o_P(1)<\eta)\\
&\quad= P(Y_1\le\mu,\ Y_2^+\le\eta)+o(1),
\end{aligned}
$$

where $\mathbf{Y}=(Y_1,Y_2)=(W_1,W_2+W_3)$ is $N(\mathbf{0},\Sigma_{\mathbf{Y}})$-distributed, with covariance matrix $\Sigma_{\mathbf{Y}}$ defined in the assertion of the theorem.

Now, assume d.f. $F(x)$ is in Case I for minima with $l_F=0$. The first part of the proof in this case is similar to that for $F(x)$ in Case III. Indeed, for any real number μ and $j_n=\lfloor\delta_n+\vartheta_n\mu\rfloor$, one uses reasoning as in (20), together with the result that $\lim_{n\to\infty}n^{1/2}(H(n/\delta_n)-H(n/j_n))/c(n/j_n)=\mu$, to obtain

$$
\left\{\vartheta_n^{-1}(M^e(c_n)-\delta_n)\le\mu\right\}=\left\{\frac{-\int_{1/\delta_n}^{n/\delta_n}U_{m_n}(s)dF^{\leftarrow}(s)}{A^e(n/\delta_n)}+o_P(1)\le\mu\right\} \qquad (25)
$$

where $A^e(s)=s^{1/2}c(s)$ and $m_n=n/F(\varepsilon_n)$ for $n\ge1$. Next, one uses reasoning as in the previous case, together with (9) and conditions (11) and $\lim_{n\to\infty}\gamma_n/n^{1/2}=0$, to show that representation (21) holds with $\tilde{\vartheta}_n\approx n^{1/2}/(\gamma_nF((c_n-\tilde{c}_n)/n))$, and that $\mathrm{I}_n=o_P(1)$ and $\mathrm{II}_n=o_P(1)$ (use part (iii) instead of part (i) of Lemma 3.4 in the reasoning to show that $\mathrm{II}_n=o_P(1)$). Thus, we have

$$
\tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n)-\delta_n)=(n-\eta_n)/(n^{1/2}/\gamma_n)+o_P(1). \qquad (26)
$$

The asymptotic analysis of $\{(n-\eta_n)/(n^{1/2}/\gamma_n)\}_{n\ge1}$ for $F(x)$ in Case I is not directly analogous to the analysis of $\{(n-\eta_n)/n^{1/2}\}_{n\ge1}$ for $F(x)$ in Case III; the previous reasoning, by itself, leads to an indeterminate form in this case, and needs to be refined. With the notation of (23), we first have for $r \geq 0$ and $l_n = \lfloor n - (n^{1/2}/\gamma_n)(r - o_P(1)) \rfloor$ that

$$
\begin{aligned}
&\{(n-\eta_n)/(n^{1/2}/\gamma_n) + o_P(1) < r\} \\
&= \left\{ \frac{\sum_{i=1}^{l_n}(W_{i,n} - EW_{i,n})}{l_n^{1/2} c(\tau_n)} \leq \frac{\tilde{c}_n - l_n EW_{i,n}}{l_n^{1/2} c(\tau_n)} \right\} \\
&= \left\{ \frac{\sum_{i=1}^{n}(W_{i,n} - EW_{i,n})}{n^{1/2} c(\tau_n)} + o_P(1) \leq r \right\} \\
&= \left\{ \frac{\sum_{i=1}^{K_n} X_i I(X_i \leq \varepsilon_n) - nEW_{i,n}}{n^{1/2} c(n/\delta_n)} + o_P(1) \leq r \right\} \\
&= \left\{ \gamma_n \left(\frac{\sum_{i=1}^{K_n}(X_i I(X_i \leq \varepsilon_n) - \phi_n)}{n^{1/2} F^{\leftarrow}(n/\delta_n)} \right) + \frac{\phi_n (K_n - m_n)}{n^{1/2} c(n/\delta_n)} + o_P(1) \leq r \right\} \\
&= \left\{ \gamma_n \left(\frac{\sum_{i=1}^{m_n}(X_i I(X_i \leq \varepsilon_n) - \phi_n)}{n^{1/2} F^{\leftarrow}(n/\delta_n)} \right) + \frac{\phi_n (K_n - m_n)}{n^{1/2} c(n/\delta_n)} + o_P(1) \leq r \right\}.
\end{aligned} \tag{27}
$$

Here, the second equality uses (17), Lemma 3.4 (ii) and a Doeblin-Anscombe type argument; and the fifth equality uses condition (12) and a variation of the Doeblin-Anscombe type argument based on the weak convergence conclusion of Lemma 3.3.

Unlike the Case III argument, further analysis in this case cannot be based on separate analysis (as in (24)) of the two summands in the last expression in (27). So proceed as follows. Denote $\tilde{S}_n := \sum_{i=1}^{m_n}(X_i I(X_i \leq \varepsilon_n) - \phi_n)/(n^{1/2} F^{\leftarrow}(n/\delta_n))$ and for each $r \geq 0$, $\xi_n = \xi_n(r) := \lfloor m_n + ((r - \gamma_n \tilde{S}_n - o_P(1))/(\phi_n/(n^{1/2} c(n/\delta_n)))) \rfloor$ for $n \geq 1$, and obtain that

$$
\begin{aligned}
&\left\{ \gamma_n \tilde{S}_n + \frac{\phi_n (K_n - m_n)}{n^{1/2} c(n/\delta_n)} + o_P(1) \leq r \right\} \\
&= \{K_n \leq \xi_n\} = \left\{ \sum_{i=1}^{\xi_n} I(X_i \leq \varepsilon_n) \geq n \right\} \\
&= \left\{ \gamma_n \frac{1}{\sqrt{m_n}} \sum_{i=1}^{\xi_n} \left(\frac{I(X_i \leq \varepsilon_n) - \tau_n}{\sqrt{\tau_n(1-\tau_n)}} \right) \geq \left(\frac{-\tau_n F^{\leftarrow}(\tau_n)}{\sqrt{1-\tau_n}\phi_n} \right)(r - \gamma_n \tilde{S}_n) + o_P(1) \right\} \\
&= \left\{ \frac{\gamma_n}{\sqrt{1-\tau_n}} \left(\frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \left(\frac{I(X_i \leq \varepsilon_n) - \tau_n}{\sqrt{\tau_n}} \right) - \left(\frac{\tau_n F^{\leftarrow}(\tau_n)}{\phi_n} \right) \tilde{S}_n \right) \geq -r + o_P(1) \right\} \\
&= \left\{ \frac{1}{\sqrt{1-\tau_n}} \left(\gamma_n \left(1 - \frac{\tau_n F^{\leftarrow}(\tau_n)}{\phi_n} \right) \left(\frac{\alpha_{m_n}(\tau_n)}{\sqrt{\tau_n}} \right) \right. \right. \\
&\qquad \left. \left. + \left(\frac{m_n}{\delta_n} \right)^{1/2} \left(\frac{\tau_n F^{\leftarrow}(\tau_n)}{\phi_n} \right) \left(\frac{\int_0^{n/\delta_n} \alpha_{m_n}(s) dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2} c(n/\delta_n)} \right) \right) \geq -r + o_P(1) \right\} \\
&= \left\{ -\frac{U_{m_n}(n/\delta_n)}{(n/\delta_n)^{1/2}} + \frac{\int_{1/\delta_n}^{n/\delta_n} U_{m_n}(s) dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2} c(n/\delta_n)} \geq -r + o_P(1) \right\}.
\end{aligned} \tag{28}
$$

Here, the third equality uses $\lim_{n\to\infty}\gamma_n/n^{1/2} = 0$; the fourth equality uses the condition $\lim_{n\to\infty}\gamma_n^2/n^{1/2} = 0$ in a variation of a Doeblin-Anscombe type argument based on $(\xi_n - m_n)/s_n \Rightarrow$ a $N(0,1)$-distributed r.v. (from Lemma 3.2); the fifth equality is the crucial step which shows how the sum can be rearranged into stable parts; and the last equality uses (14) and the limits $\lim_{n\to\infty}\tau_n F^{\leftarrow}(\tau_n)/\phi_n = 1$ and $\lim_{n\to\infty}\gamma_n((\tau_n F^{\leftarrow}(\tau_n)/\phi_n) - 1) = 1$ (see (9)).

Finally, we have from (25)-(28), that for each real number μ and each $\eta \geq 0$,

$$
\begin{aligned}
&P(\vartheta_n^{-1}(M^e(c_n) - \delta_n) \leq \mu,\, \tilde{\vartheta}_n^{-1}(M^{\pi_n}(c_n) - \delta_n) < \eta) \\
&= P\left(\frac{-\int_{1/\delta_n}^{n/\delta_n} U_{m_n}(s)dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2}c(n/\delta_n)} + o_P(1) \leq \mu,\, \frac{U_{m_n}(n/\delta_n)}{(n/\delta_n)^{1/2}} - \frac{\int_{1/\delta_n}^{n/\delta_n} U_{m_n}(s)dF^{\leftarrow}(s)}{(n/\delta_n)^{1/2}c(n/\delta_n)} + o_P(1) \leq \eta\right) \\
&= P(Y_1 \leq \mu,\, Y_2^+ \leq \eta) + o(1),
\end{aligned}
$$

where $\mathbf{Y} = (Y_1, Y_2)$ is $N(\mathbf{0}, \Sigma_{\mathbf{Y}})$-distributed with covariance matrix $\Sigma_{\mathbf{Y}}$ defined in the assertion of the theorem. □

4 Results for large-sized budget constraints

In this section, *the sequence of budget constraints* $(c_n)_{n\geq 1}$ *is assumed to satisfy the large-sized budget constraint property* $\lim_{n\to\infty} n^{-1/2}(c_n - n\theta) = 0$, for some $0 < \theta < EX_1$, where $X_1, X_2, \cdots$ is the sequence of salary demands (nonnegative i.i.d. r.v.'s with d.f. $F(x)$ and $l_F = 0$). The parameters $(\tilde{c}_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ associated with the two-stage threshold policies $(\pi_n)_{n\geq 1}$ of the Introduction are given as follows. The sequence of the first-stage budget constraints $(\tilde{c}_n)_{n\geq 1}$ are assumed to satisfy $0 < \tilde{c}_n < c_n$ for all $n \geq 1$ and $\lim_{n\to\infty} n^{-1/2}(\tilde{c}_n - n\theta) = 0$; and the first-stage thresholds $(\varepsilon_n)_{n\geq 1}$ are given by $\varepsilon_n := \varepsilon = F^{\leftarrow}(H^{\leftarrow}(\theta))$ for all $n \geq 1$. The location (centrality) parameters for the on-line and off-line interview counts in the convergence theorem are the constants $\delta_n := n/H^{\leftarrow}(\theta)$ for $n \geq 1$. Observe that under the continuity assumption of the next theorem, the parameters θ, ε and $(\delta_n)_{n\geq 1}$ are related by $\theta = H(F(\varepsilon)) = E(X_1|X_1 \leq \varepsilon) = H(n/\delta_n)$ and $H^{\leftarrow}(\theta) = F(\varepsilon) = n/\delta_n$; in particular, $\theta \leq \varepsilon$. This continuity assumption on $F(x)$ ensures that condition (11) holds, and the convergence assumptions on $(c_n)_{n\geq 1}$ and $(\tilde{c}_n)_{n\geq 1}$ imply that $\lim_{n\to\infty} n^{1/2}(1 - (\tilde{c}_n/c_n)) = 0$ (as in (13)). As in Theorem 2.2, the inequality $M^e(c_n) \leq M^{\pi_n}(c_n)$ carries over (with probability one) to the limit r.v.'s as $\mathcal{M}^\pi \geq 0$ since $\lim_{n\to\infty} F((c_n - \tilde{c}_n)/n) = 0$.

Theorem 4.1 *If $F(x)$ is continuous and strictly increasing on its support, $l_F = 0$ and $EX_1^2 < \infty$, then*

$$\left(n^{-1/2}(M^e(c_n) - \delta_n),\, F((c_n - \tilde{c}_n)/n)n^{-1/2}(M^{\pi_n}(c_n) - \delta_n)\right) \Rightarrow (\mathcal{M}^e, \mathcal{M}^\pi)$$

where $(\mathcal{M}^e, \mathcal{M}^\pi) = (Y_1, Y_2^+)$ and $\mathbf{Y} = (Y_1, Y_2)$ is $N(\mathbf{0}, \Sigma(\theta))$-distributed. The covariance matrix $\Sigma(\theta)$ is a symmetric matrix $\Sigma(\theta) = (\Sigma_{i,j})_{i=1,2;j=1,2}$ given by

$$\Sigma_{1,1} = \frac{\sigma^2(\theta)}{(F(\varepsilon))^3(\varepsilon - \theta)^2},$$

$$
\begin{aligned}
\Sigma_{1,2} &= \left(\frac{\varepsilon-\theta}{\theta}\right)\left(\left(\frac{\sigma(\theta)}{(\varepsilon-\theta)F(\varepsilon)}\right)^2 - \left(\frac{1-F(\varepsilon)}{F(\varepsilon)}\right)\right), \textit{ and} \\
\Sigma_{2,2} &= \theta^{-2}\left(\frac{\sigma^2(\theta)}{F(\varepsilon)} - (\varepsilon-\theta)^2(1-F(\varepsilon))\right) \textit{ where} \\
\sigma^2(\theta) &= \int_0^{F(\varepsilon)}\int_0^{F(\varepsilon)} (s\wedge t - st)dF^{\leftarrow}(s)dF^{\leftarrow}(t).
\end{aligned}
$$

There are two additional points to mention concerning the statement of Theorem 4.1. First, a more direct analogue of Theorem 2.2 would replace $n^{-1/2}(M^e(c_n)-\delta_n)$ by $F(\varepsilon)n^{-1/2}(M^e(c_n)-\delta_n) = H^{\leftarrow}(\theta)n^{-1/2}(M^e(c_n)-\delta_n)$ and replace (Y_1, Y_2^+) by $(F(\varepsilon)Y_1, Y_2^+)$. Second, it is curious that the variances of $F(\varepsilon)Y_1$ and Y_2 have the representations

$$
\begin{aligned}
\mathrm{Var}(F(\varepsilon)Y_1) &= (F(\varepsilon))^2\Sigma_{1,1} = \frac{\mathrm{Var}(\varepsilon - X_1|X_1\le\varepsilon)}{(E(\varepsilon-X_1|X_1\le\varepsilon))^2} + P(X_1>\varepsilon) \text{ and} \\
\mathrm{Var}(Y_2) &= \Sigma_{2,2} = \mathrm{Var}(X_1|X_1\le\varepsilon)/(E(X_1|X_1\le\varepsilon))^2;
\end{aligned}
$$

and $EY_2^+ = (2\pi)^{-1/2}\Sigma_{2,2}^{1/2}$ and $\mathrm{Var}\,Y_2^+ = ((\pi-1)/(2\pi))\Sigma_{2,2}$.

Proof of Theorem 4.1. The proof goes along the same lines as the proof of Theorem 2.2 for $F(x)$ in Case III and uses standard approximation arguments e.g. given by Csörgő, Csörgő and Horváth [7, Chapter 10]. First, observe that for any real number μ, and $j_n = \lfloor n^{1/2}\mu + \delta_n\rfloor$

$$
\{n^{-1/2}(M^e(c_n)-\delta_n)\le\mu\} = \left\{-\int_{1/\delta_n}^{F(\varepsilon)} U_{j_n}(s)dF^{\leftarrow}(s) + o_P(1) \le (F(\varepsilon))^{3/2}(\varepsilon-\theta)\mu\right\}. \quad (29)
$$

To justify (29), note that

$$
\begin{aligned}
& j_n^{-1/2}\left(c_n - j_n\int_0^{n/j_n} F^{\leftarrow}(s)ds\right) \\
&= j_n^{-1/2}\left(\delta_n\int_{n/j_n}^{n/\delta_n} F^{\leftarrow}(s)ds + (\delta_n - j_n)\int_0^{n/j_n} F^{\leftarrow}(s)ds\right) + o(1) \\
&=: K_1 + K_2 + o(1).
\end{aligned}
$$

It is easy to check that $K_1 = (F(\varepsilon))^{3/2}\varepsilon\mu + o(1)$ and $K_2 = -(F(\varepsilon))^{3/2}\theta\mu + o(1)$.

For the convergence analysis of $M^{\pi_n}(c_n)$, argue as in the proof of Theorem 2.2 to obtain

$$
F((c_n - \tilde{c}_n)/n)n^{-1/2}(M^{\pi_n}(c_n) - \delta_n) = n^{-1/2}(n-\eta_n) + o_P(1),
$$

where the r.v. η_n is defined in (3), and for $\tau\ge 0$ and $l_n = \lfloor n - n^{1/2}(\tau - o_P(1))\rfloor$

$$
\begin{aligned}
& \{n^{-1/2}(n-\eta_n) + o_P(1) < \tau\} = \left\{\sum_{i=1}^{l_n} W_{i,n} \le \tilde{c}_n\right\} \\
&= \left\{n^{-1/2}\left(\sum_{i=1}^{n} W_{i,n} - n\theta\right) + o_P(1) \le n^{-1/2}(\tilde{c}_n - l_n\theta)\right\} \\
&= \left\{\theta^{-1}n^{-1/2}\sum_{i=1}^{m_n}(X_iI(X_i\le\varepsilon) - \phi) + (K_n - m_n)/(n^{1/2}/F(\varepsilon)) + o_P(1) \le \tau\right\}.
\end{aligned}
$$

Here, $K_n := k_{1,n} + \cdots + k_{n,n}$, $m_n := EK_n = n/F(\varepsilon)$ and $\phi := EX_iI(X_i \leq \varepsilon) = F(\varepsilon)\theta$. To settle the joint convergence of $M^e(c_n)$ and $M^{\pi_n}(c_n)$ obtain for any real numbers μ, z_1 and z_2

$$P\left(n^{-1/2}(M^e(c_n) - \delta_n) \leq \mu, F(\varepsilon)n^{-1/2}(K_n - m_n) \leq z_1, \theta^{-1}n^{-1/2}\sum_{i=1}^{m_n}(X_iI(X_i \leq \varepsilon) - \phi) \leq z_2\right)$$

$$= P\left(-\int_{1/\delta_n}^{F(\varepsilon)} U_{m_n}(s)dF^{\leftarrow}(s) \leq (F(\varepsilon))^{3/2}(\varepsilon - \theta)\mu,\ -(F(\varepsilon))^{-1/2}U_{m_n}(F(\varepsilon)) \leq z_1,\right.$$

$$\left.\theta^{-1}(F(\varepsilon))^{-1/2}\left(\varepsilon U_{m_n}(F(\varepsilon)) - \int_{1/\delta_n}^{F(\varepsilon)} U_{m_n}(s)dF^{\leftarrow}(s)\right) \leq z_2\right) + o(1).$$

Proceed as in the proof of Theorem 2.2 to complete the proof of the theorem. □

Corollary 4.2 *If $F(x)$ is continuous and strictly increasing on its support, $l_F = 0$, and $EX_1^2 < \infty$, then*
(i) the sequence of two-stage threshold policies $(\pi_n)_{n\geq 1}$ is a consistent approximator of the off-line interview-count strategy τ^e;
(ii) $F((c_n - \tilde{c}_n)/n)(M^{\pi_n}(c_n) - \delta_n)/(M^e(c_n) - \delta_n) \Rightarrow \mathcal{M}^\pi/\mathcal{M}^e$ with $\mathcal{M}^e$ and $\mathcal{M}^\pi$ as given in Theorem 4.1; and
(iii) if $\lim_{n\to\infty} n^{1/2}F((c_n - \tilde{c}_n)/n) = \infty$, then $M^{\pi_n}(c_n)/M^e(c_n) \to 1$ in probability as $n \to \infty$.

A Appendix

Recall the following definitions, relations, and results concerned with domains of attraction for minima.

Let $X_1, X_2, \cdots$ be i.i.d. r.v.'s with d.f. F. $F(x)$ is said to be in the domain of attraction of d.f. $K(x)$ for minima if there exists constants $(a_n)_{n\geq 1}$, $a_n > 0$ and $(b_n)_{n\geq 1}$, $b_n \in \mathbb{R}$, such that

$$P\left(a_n\left(\min_{1\leq i\leq n} X_i - b_n\right) \leq x\right) \to K(x) \text{ for all continuity points } x \text{ of } K. \tag{30}$$

In this setting we say F is in Case I, in Case II with parameter $\alpha > 0$ or in Case III with parameter $\alpha > 0$, if the limit d.f. K is respectively given by $K_{\mathrm{I}}(x) = 1 - \exp(-e^x)$ for $x \in \mathbb{R}$; by $K_{\mathrm{II}}^\alpha(x) = 1 - \exp(-(-x)^{-\alpha})$ for $x < 0$; or by $K_{\mathrm{III}}^\alpha(x) = 1 - \exp(-x^\alpha)$ for $x > 0$.

Let $Y_i = -X_i$ for $i \geq 1$, so that $Y_1, Y_2, \cdots$ are i.i.d. r.v.'s with d.f. $G(x) = 1 - F((-x)-)$, and $r_G := \sup\{x : G(x) < 1\} = -l_F$. F is in the domain of attraction for minima in Case I, II or III iff respectively G is in the domain of attraction for maxima associated with d.f.'s Λ, Φ_α, or Ψ_α. See Resnick [12] and de Haan [10] for results on domains of attraction for maxima and definitions of these d.f.'s. In the following paragraphs we list some properties in each of the Cases I, II, and III which are used in this paper.

If F is in Case I for minima, then constants $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are given by $b_n = F^{\leftarrow}(1/n)$ and $a_n^{-1} = g(b_n)$ for $n \geq 1$, for example with $g(t) = \int_{l_F}^t F(x)dx/F(t)$. Function $F(x)$ satisfies $\lim_{t\downarrow 0} F(t - xg(t))/F(t) = e^{-x}$ for all $x \in \mathbb{R}$, and has representation $F(x) = \tilde{k}(x)\exp\left(-\int_x^{\tilde{z}} \tilde{h}(t)/\tilde{g}(t)\,dt\right)$ for $0 < x < \tilde{z}$, for some $\tilde{z} > 0$ and measurable functions $\tilde{k}$, $\tilde{h}$ and $\tilde{g}$ on $(0, \tilde{z})$, with $\tilde{k}(x) \to k > 0$, $\tilde{h}(x) \to 1$, and $(\tilde{g})'(x) \to 0$ as $x \to 0$, and $\tilde{g} > 0$ and is absolutely continuous on $(0, \tilde{z})$ with $\tilde{g}(t) \approx g(t)$ as $t \to 0$ (see e.g. Corollary 1.7 of [12]).

The inverse function $F^{\leftarrow}(s)$ is slowly varying at zero, and satisfies

$$\lim_{t\downarrow 0}\frac{F^{\leftarrow}(tx)-F^{\leftarrow}(tz)}{F^{\leftarrow}(ty)-F^{\leftarrow}(tw)}=\frac{\log\, x-\log\, z}{\log\, y-\log\, w}$$

for all $0<x,y,w,z<\infty$ with $y\neq w$. The function $c(s)$ is defined in Section 2 as $c(s):=s^{-1}\int_0^s udF^{\leftarrow}(u)=F^{\leftarrow}(s)-H(s)$. The function $c(s)$ satisfies the following properties used in this paper (see e.g. [10] and [11]): $c(s)>0$; $c(s)$ is slowly varying at zero; and if l_F is finite, then $\lim_{s\downarrow 0}c(s)=0$. There exists a finite constant k such that for $0<s\leq 1/2$, $F^{\leftarrow}(s)=k+c(s)-\int_s^1 u^{-1}c(u)du$ and $H(s)=k-\int_s^1 u^{-1}c(u)du$. As $s\downarrow 0$, $c(s)/F^{\leftarrow}(s)\to 0$, and $(F^{\leftarrow}(s)-F^{\leftarrow}(sx))/c(s)\to -\log\, x$ for all $x>0$; and if $l_F=0$, then $c(s)/g(F^{\leftarrow}(s))\to 1$, $F(F^{\leftarrow}(s)+zc(s))/s\to e^z$, and $s^{-1}\int_0^s(F^{\leftarrow}(u))^2du-\left(s^{-1}\int_0^s F^{\leftarrow}(u)du\right)^2\approx(c(s))^2$.

If F is in Case II for minima, with $\alpha>0$, and $a=1/\alpha$, then $a_n^{-1}=-F^{\leftarrow}(1/n)$ and $b_n=0$ for $n\geq 1$, $l_F=-\infty$ and $\lim_{n\to\infty}nF(a_n^{-1}x)=(-x)^{-\alpha}$ for $x<0$. The function $F(-x)$ is regularly varying as $x\to\infty$ with index $-\alpha=-1/a$; and $F^{\leftarrow}(s)=-s^{-a}L(s)$ where L is slowly varying at zero.

If F is in Case III for minima, with $\alpha>0$, and $a=-1/\alpha$, then $a_n^{-1}=F^{\leftarrow}(1/n)-l_F$ and $b_n=l_F$ for $n\geq 1$, l_F is finite and $\lim_{n\to\infty}nF(a_n^{-1}x+l_F)=x^{\alpha}$ for $x>0$. The function $F(l_F+x^{-1})$ is regularly varying as $x\to\infty$ with index $-\alpha=1/a$; and $F^{\leftarrow}(s)=l_F+s^{-a}L(s)$ where L is slowly varying at zero.

Finally we state two results concerning functions which are slowly varying at zero, which we frequently use in this paper.

- (Karamata's Theorem (see [8, Lemma 1]).) Let $L(x)$ be slowly varying at zero. If $\beta<1$, then
$$\lim_{s\downarrow 0}\int_0^s u^{-\beta}L(u)du/\left(s^{1-\beta}L(s)\right)=\frac{1}{1-\beta}.$$
- (Karamata's Representation (see e.g. [12, Section 0.4]).) For $0<t<t_0$, $L(t)$ is slowly varying at zero if and only if $L(t)=k(t)\exp\left(\int_t^{t_0}\varepsilon(u)/u\, du\right)$ for some measurable functions $k:(0,t_0)\to\mathbb{R}_+$ and $\varepsilon:(0,t_0)\to\mathbb{R}$ satisfying $\lim_{t\downarrow 0}k(t)=k_0$ for some constant $k_0>0$ and $\lim_{t\to 0}\varepsilon(t)=0$.

Acknowledgment. The authors are grateful to the School of Mathematics of the Georgia Institute of Technology in Atlanta, the Econometric Institute of the Erasmus University Rotterdam and the Stieltjes Institute in The Netherlands for reciprocal invitations to pursue this research, and for their hospitality.

References

[1] Boshuizen, F.A. and Kertz, R.P. (1995) Smallest-fit selection of random sizes under a sum constraint: weak convergence and moment comparisons, *preprint.*

[2] Chen, R.W., Nair, V.N., Odlyzko, A.M., Shepp, L.A. and Vardi, Y. (1984) Optimal sequential selection of N random variables under a constraint. *J. Appl. Prob.* **21**, 537-547.

[3] Chow, Y.S. and Teicher, H. (1978) *Probability Theory: Independence, Interchangeability, Martingales.* Springer-Verlag, New York.

[4] Chung, K.L. (1974) *A Course in Probability Theory, 2nd edition.* Academic Press, New York.

[5] Coffman, E.G., Flatto, L. and Weber, R.R. (1987) Optimal selection of stochastic intervals under a sums constraint. *Adv. Appl. Prob.* **19**, 454-473.

[6] Csörgő, M., Csörgő, S., Horváth, L. and Mason, D.M. (1986) Weighted empirical and quantile processes. *Ann. Probab.* **14**, 31-85.

[7] Csörgő, M., Csörgő, S. and Horváth, L. (1986) *An Asymptotic Theory for Empirical Reliability and Concentration Processes.* Lecture Notes in Statistics **33**. Springer-Verlag, Berlin.

[8] Csörgő, S. and Mason, D.M. (1986) The asymptotic distribution of sums of extreme values from a regularly varying distribution. *Ann. Probab.* **14**, 974-983.

[9] Csörgő, S., Haeusler, E. and Mason, D. (1991) The asymptotic distribution of extreme sums. *Ann. Probab.* **19**, 783-811.

[10] Haan, L. de (1970) *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*, Mathematical Centre Tract **32**, Mathematical Centre, Amsterdam.

[11] Lo, G.S. (1989) A note on the asymptotic normality of extreme values. *J. Statist. Planning and Inference* **22**, 127-136.

[12] Resnick, S. (1987) *Extreme Values, Regular Variation, and Point Processes.* Springer-Verlag, New York.

[13] Rhee, W. and Talagrand, M. (1991) A note on the selection of random variables under a sum constraint. *J. Appl. Prob.* **28**, 919-923.